ON THE BEHAVIOR AT INFINITY OF AN INTEGRABLE FUNCTION

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We denote by x a real variable and by n a positive integer variable. The reference measure on the real line $\mathbb{R}$ is the Lebesgue measure. In this note we will use only basic properties of the Lebesgue measure and integral on $\mathbb{R}$.

It is well known that the fact that a function tends to zero at infinity is a condition neither necessary nor sufficient for this function to be integrable. However, we have the following result.

Theorem 1. *Let f be an integrable function on the real line $\mathbb{R}$. For almost all $x \in \mathbb{R}$, we have*

$$\lim_{n\to\infty} f(nx) = 0 \ . \tag{1}$$

Remark 1. It is too much hope in Theorem 1 for a result *for all x* because we consider an integrable function f, which can take arbitrary values on a set of zero measure. Even if we consider only continuous functions, the result does not hold for all x. Indeed a classical result, using a Baire category argument, tells us that if f is a continuous function on $\mathbb{R}$ such that for all nonzero x, $\lim_{n\to\infty} f(nx) = 0$, then $\lim_{x\to\pm\infty} f(x) = 0$. Thus for a continuous integrable function f which does not tend to zero at infinity, property (1) is true for almost all x and not for all x.

Remark 2. Let f be an integrable and nonnegative function on $\mathbb{R}$. We have $\int f(nx)\ \mathrm{d}x = \frac{1}{n}\int f(x)\ \mathrm{d}x$. Hence for any nonnegative real sequence (ε_n) such that $\sum_n \epsilon_n/n < +\infty$, we have

$$\sum_n \int \varepsilon_n f(nx)\ \mathrm{d}x < +\infty \ ,$$

and the monotone convergence theorem (or Fubini's theorem) ensures that the function $x \mapsto \sum_n \varepsilon_n f(nx)$ is integrable, hence almost everywhere finite. In particular, for almost all x, we have $\lim_{n\to\infty} \varepsilon_n f(nx) = 0$. This argument is not sufficient to prove Theorem 1.

Now we will state that, in a sense, Theorem 1 gives an optimal result. The strength of the following theorem lies in the fact that the sequence (a_n) can tend to infinity arbitrarily slowly.

Theorem 2. *Let (a_n) be a real sequence which tends to $+\infty$. There exists a continuous and integrable function f on $\mathbb{R}$ such that, for almost all x,*

$$\limsup_{n\to\infty} a_n f(nx) = +\infty \ .$$

Moreover, there exists an integrable function f on $\mathbb{R}$ such that, for all x,

$$\limsup_{n\to\infty} a_n f(nx) = +\infty \ .$$

Question. Under the hypothesis of Theorem 2, does there exist a *continuous* and integrable f such that, for *all* x, $\limsup_{n\to\infty} a_n f(nx) = +\infty$?
We do not know the answer to this question, and we propose it to the reader. However, the next remark shows that the answer is positive under a slightly more demanding hypothesis.

Remark 3. If the sequence (a_n) is nondecreasing and satisfies $\sum_n \frac{1}{na_n} < +\infty$, then there exists a continuous and integrable function f on $\mathbb{R}$ such that for *all* x, $\limsup_{n\to\infty} a_n f(nx) = +\infty$.

Remark 4. In Theorem 2 we cannot replace the hypothesis $\lim_n a_n = +\infty$ by $\limsup_n a_n = +\infty$. Indeed, by a simple change of variable we can deduce from Theorem 1 the following result: for all integrable functions f on $\mathbb{R}$,

$$\lim_{n\to\infty} nf(n^2x) = 0 \quad \text{for almost all } x.$$

(Apply Theorem 1 to the function $x \mapsto xf(x^2)$.) Thus the conclusion of Theorem 2 is false for the sequence (a_n) defined by

$$a_n = \begin{cases} \sqrt{n} & \text{if } n \text{ is a square of integer,} \\ 0 & \text{if not.} \end{cases}$$

In the remainder of this note, we give proofs of the two theorems and of Remark 3.

Proof of Theorem 1. The function f is integrable on $\mathbb{R}$. Let us fix $\varepsilon > 0$ and denote by E the set of points $x > 0$ such that $|f(x)| \geq \varepsilon$. We know that E has finite measure. We are going to show that, for almost all $x \in [0,1]$, we have $nx \in E$ for only finitely many n's. (If A is a measurable subset of $\mathbb{R}$, we denote by $|A|$ its Lebesgue measure.)

For each integer $m \geq 1$, let $E_m := E \cap (m-1, m]$. Let us fix $a \in (0,1)$. For each integer $n \geq 1$, we consider the set

$$F_n := \left(\frac{1}{n}E\right) \cap [a,1) = \left(\frac{1}{n}\bigcup_{m\geq 1} E_m\right) \cap [a,1) = \frac{1}{n}\bigcup_{m\geq 1}\left(E_m \cap [na, n)\right) \ .$$

We have

$$\sum_{n=1}^{+\infty} |F_n| = \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} \frac{1}{n}\left|E_m \cap [na,n)\right| \ .$$

In this doubly indexed sum of positive numbers, we can invert the order of summation. Moreover, noticing that $E_m \cap [na, n) = \emptyset$ if $n > m/a$ or $n \leq m-1$, we obtain

$$\sum_{n=1}^{+\infty} |F_n| = \sum_{m=1}^{+\infty}\sum_{n=m}^{[m/a]} \frac{1}{n}\left|E_m \cap [na,n)\right| \leq \sum_{m=1}^{+\infty} |E_m| \sum_{n=m}^{[m/a]} \frac{1}{n} \ .$$

By comparison of the discrete sum with an integral, we see that, for all $m \geq 1$, $\sum_{n=m}^{[m/a]} \frac{1}{n} \leq (1 - \ln a)$. Thus we have

$$\sum_{n=1}^{+\infty} |F_n| \leq (1 - \ln a) \sum_{m=1}^{+\infty} |E_m| = (1 - \ln a)|E| < +\infty .$$

This implies that almost every x belongs to only finitely many sets F_n. (This statement is the Borel-Cantelli lemma, which has a one line proof :
$\sum \mathbb{1}_{F_n} < +\infty$ almost everywhere since

$$\int \sum \mathbb{1}_{F_n}(x) \, \mathrm{d}x = \sum \int \mathbb{1}_{F_n}(x) < +\infty .)$$

Returning to the definition of F_n, we conclude that, for almost all $x \in [a, 1]$, for all large enough n , $x \notin F_n$, i.e. $nx \notin E$. Since a is arbitrary, we have in fact: for almost all $x \in [0, 1]$, for all large enough n, $nx \notin E$.

We have proved that, for all $\varepsilon > 0$, for almost all $x \in [0, 1]$, for all large enough n, $|f(nx)| \leq \varepsilon$. Since we have to consider only countably many ε's, we can invert *for all* $\varepsilon > 0$ and *for almost all* $x \in [0, 1]$. We conclude that, for almost all $x \in [0, 1]$, $\lim_{n\to\infty} f(nx) = 0$. It is immediate, by a linear change of variable (for example), that this result extends to almost all $x \in \mathbb{R}$. □

Proof of Theorem 2. We will utilize the following theorem, a fundamental result in the metric theory of Diophantine approximation [1, Theorem 32].

Khinchin's Theorem. *Let (b_n) be a sequence of positive real numbers such that the sequence (nb_n) is nonincreasing and the series $\sum_n b_n$ diverges. For almost all real numbers x, there are infinitely many integers n such that* $\operatorname{dist}(nx, \mathbb{Z}) < b_n$.

We will also make use of the following lemma, which will be proved in the sequel.

Lemma 1. *Let (c_n) be a sequence of nonnegative real numbers going to zero. There exists a sequence of positive real numbers (b_n) such that the sequence (nb_n) is nonincreasing, $\sum_n b_n = +\infty$, and $\sum_n b_n c_n < +\infty$.*

Let us prove Theorem 2.

Replacing if necessary a_n by $\inf_{k\geq n} a_k$, we can suppose that the sequence (a_n) is nondecreasing. Applying the preceding lemma to the the sequence $c_n = 1/\sqrt{a_n}$, we obtain a sequence (b_n) such that the sequence (nb_n) is nonincreasing, $\sum_n b_n = +\infty$, and $\sum_n b_n/\sqrt{a_n} < +\infty$. The sequence (b_n) tends to zero, and we can impose the additional requirement that $b_n < 1/2$ for all n.

We consider the function f_1 defined on $\mathbb{R}$ by

$$f_1(x) = \begin{cases} 1/\sqrt{a_n} & \text{if } |x - n| \leq b_n \text{ for an integer } n \geq 1, \\ 0 & \text{if not.} \end{cases}$$

This function is integrable, due to the last condition imposed on (b_n).

By Khinchin's theorem, for almost all $x > 0$, there exist pairs of positive integers $(n, k(n))$, with arbitrarily large n, such that

$$|nx - k(n)| \leq b_n.$$

Let us consider one fixed such x in the interval $(0, 1)$. We have $\lim_{n\to\infty} k(n) = +\infty$ and, since $\lim_{n\to+\infty} b_n = 0$, we have $k(n) \leq n$ for all large enough n. For such an

n, we have

$$|nx - k(n)| \le b_{k(n)} \quad \text{and hence} \quad f_1(nx) = \frac{1}{\sqrt{a_{k(n)}}} \; .$$

(We used here the fact that the sequence (b_n) is nonincreasing.) Thus, for arbitrarily large n, we have

$$a_n f_1(nx) = \frac{a_n}{\sqrt{a_{k(n)}}} \ge \sqrt{a_{k(n)}} \; .$$

(We used here the fact that the sequence (a_n) is nondecreasing.) This proves that $\limsup_{n\to\infty} a_n f_1(nx) = +\infty$. This argument applies to almost all x between 0 and 1.

For each integer $m \ge 1$, let us denote by f_m the function $f_m(x) = f_1(x/m)$. This function f_m is nonnegative and integrable on $\mathbb{R}$. It is locally a step function. For almost all x between 0 and m, we have

$$\limsup_{n\to\infty} a_n f_m(nx) = +\infty \; .$$

From this, it is not difficult to construct a continuous and integrable function f on $\mathbb{R}$ such that, for all $m > 0$, there exists $A_m > 0$ with $f \ge f_m$ on $[A_m, +\infty)$. (For example, we can choose an increasing sequence of numbers (A_m) such that

$$\int_{A_m}^{+\infty} f_1(x) + f_2(x) + \cdots + f_m(x) \; \mathrm{d}x \le \frac{1}{m^2} \; ;$$

then we define $g = f_1 + f_2 + \cdots + f_m$ on the interval $[A_m, A_{m+1})$. Since

$$\sum_m \int_{A_m}^{A_{m+1}} f_1(x) + f_2(x) + \cdots + f_m(x) \; \mathrm{d}x < \infty \; ,$$

this function g is integrable. Then we just have to find a continuous and integrable function f which dominates g; this can be achieved since the function g is locally a step function: choose f to be zero on $(-\infty, 0]$ and continuous on $\mathbb{R}$ such that $g \le f$ and, for all $m > 0$, $\int_{m-1}^{m} f(x) - g(x) \; \mathrm{d}x \le 1/m^2$, so that $\int_0^{+\infty} f(x) - g(x) \; \mathrm{d}x < +\infty$.)

For almost all $x \ge 0$, we have $\limsup_{n\to\infty} a_n f(nx) = +\infty$. A symmetrization procedure extends this property to almost all real numbers.

The first part of Theorem 2 is proved. The second part is a direct consequence. We consider the function f constructed above, and we denote by F the set of x such that the sequence $(a_n f(nx))$ is bounded. The set $\{nx \mid x \in F, n \in \mathbb{N}\}$ has zero measure. We modify the function f on this set, choosing for example the value 1. The new function is integrable and satisfies, for all x, $\limsup_{n\to\infty} a_n f(nx) = +\infty$. □

Proof of Lemma 1. The sequence (c_n) is given, and it goes to zero. We will construct by induction an increasing sequence of integers (n_k) and a nonincreasing sequence of positive numbers (d_k), and we will define $b_n = d_k/n$ for $n_{k-1} \le n < n_k$. The numbers d_k will be chosen so that $\sum_{i=n_{k-1}}^{n_k-1} b_i = 1$; thus we require that

$$d_k := \left(\sum_{i=n_{k-1}}^{n_k-1} \frac{1}{i} \right)^{-1} \; .$$

We start from $n_0 = 1$, and then we choose $n_1 > n_0$ such that, for all $n \geq n_1$, $|c_n| \leq 1/2$. In the next step, we choose $n_2 > n_1$ such that $d_2 \leq d_1$ and, for all $n \geq n_2$, $|c_n| \leq 1/4$.
More generally, if $n_1, n_2, \ldots, n_{k-1}$ have been constructed, we choose $n_k > n_{k-1}$ such that $d_k \leq d_{k-1}$ and, for all $n \geq n_k$, $|c_n| \leq 2^{-k}$. (Of course, this is possible because $\lim_{n\to+\infty} \left(\sum_{i=n_{k-1}}^{n} \frac{1}{i}\right)^{-1} = 0$.)

This defines the sequence (b_n) by blocks. The sequence (nb_n) is nonincreasing and, for all $k \geq 1$, we have

$$\sum_{i=n_{k-1}}^{n_k-1} b_i = 1 \quad \text{and} \quad \sum_{i=n_{k-1}}^{n_k-1} b_i c_i \leq 2^{1-k} .$$

This guarantees that $\sum_n b_n = +\infty$ and $\sum_n b_n c_n < +\infty$. The lemma is proved. $\square$

About Remark 3. Dirichlet's lemma in Diophantine approximation (based on the pigeon-hole principle) concerns the particular case $b_n = 1/n$ in Khinchin's theorem and it gives a result *for all* x.

Lemma 2 (Dirichlet's Lemma)**.** *For all real numbers x, there exist infinitely many integers n such that* $\mathrm{dist}(nx, \mathbb{Z}) \leq \frac{1}{n}$.

Now, we justify Remark 3. We consider a nondecreasing sequence of positive real numbers (a_n) such that

$$\sum_n \frac{1}{na_n} < +\infty .$$

We claim that there exists a sequence of positive real numbers (b_n) such that

$$b_n a_n \to +\infty \quad \text{and} \quad \sum \frac{b_n}{n} < +\infty .$$

Here is a proof of this claim: for each $k \geq 1$, there exists $n(k)$ such that

$$\sum_{n\geq n(k)} \frac{1}{na_n} \leq \frac{1}{k^2}.$$

We have

$$\sum_n \mathrm{card}\{k \mid n(k) \leq n\} \frac{1}{na_n} = \sum_{k\geq 1} \sum_{n\geq n(k)} \frac{1}{na_n} < +\infty ,$$

and we can define $b_n := \mathrm{card}\{k \mid n(k) \leq n\}/a_n$.

Given this sequence (b_n), we consider the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} b_k & \text{if } |x-k| \leq 1/k,\ k \text{ an integer}, k \geq 2, \\ 0 & \text{if not.} \end{cases}$$

This function is integrable.

Using Dirichlet's lemma, we have the following: for each fixed x in $(0,1)$, there exist pairs of positive integers $(n, k(n))$, with n arbitrarily large, such that $|nx - k(n)| \leq 1/n$. We have $\lim_{n\to\infty} k(n) = +\infty$ and, for all large enough n, $k(n) \leq n$. Hence there exist infinitely many n's such that

$$|nx - k(n)| \leq \frac{1}{k(n)} \quad \text{and so} \quad f(nx) = b_{k(n)} .$$

For such an n, we have

$$a_n f(nx) = a_n b_{k(n)} \geq a_{k(n)} b_{k(n)} \ .$$

(We used here the fact that the sequence (a_n) is nondecreasing.) This proves that $\limsup_{n\to\infty} a_n f(nx) = +\infty$. This result obtained for all numbers x between 0 and 1 extends to all real numbers by the same argument as the one used in the proof of Theorem 2. We can also replace the local step function by a continuous one as we did before. □

Theorem 1 answers a question asked by Aris Danilidis.

References

[1] A. Ya. Khinchin, *Continued Fractions*, Dover, Mineola, NY, 1997; reprint of (trans. Scripta Technica, Inc.) University of Chicago Press, 1961; reprint of 3rd Russian ed., State Publishing House of Physical-Mathematical Literature, Moscow, 1961.

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